Repeated Root and Common Root

1. Prove that $x^3 + ax + b = 0$ has 3 distinct roots if and only if $27b^2 + 4a^3 < 0$.

2. Prove that ω, the complex root of unity, is a repeated root of $3x^5 + 2x^4 + x^3 - 6x^2 - 5x - 4 = 0$, and hence solve the equation.

3. Prove that the equation $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} = 0$ has no repeated roots.

4. If $a_1 < a_2 < \dots < a_6$, prove that the equation $(x - a_1)(x - a_3)(x - a_5) + k^2(x - a_2)(x - a_4)(x - a_6) = 0$ has 3 distinct roots for any real k.

5. If the equation $x^4 - 4ax^3 + 6x^2 + 1 = 0$ has a repeated root q, show that $a = \frac{q^2 + 3}{3q}$.

Hence or otherwise, prove that there is only one positive a giving a repeated root, and that this value of

a is .

6. If the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0$ has a multiple root α, prove that α is a root of the equation $p_1x^{n-1} + 2p_2x^{n-2} + 3p_3x^{n-3} + \dots + (n - 1)p_{n-1}x + np_n = 0$.

7. Prove that

(a) If h is an s-multiple root of $a'(x)$ and if $a(h) = 0$, then h is an $(s + 1)$-multiple root of $a(x)$. Prove also the converse.

(b) $ax^2 + bx + c = 0$ has a double root if and only if its discriminant $\Delta = b^2 - 4ac$ is 0.

(c) If $ax^3 + 3bx^2 + 3cx + d$ has a triple root h, then $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = -h$.

(d) If $ax^3 + 3bx^2 + 3cx + d$ has a non-zero double root k, then:

(i) $ad \neq bc$

(ii) $bk^2 + 2ck + d = 0$

(iii) $k = \frac{1}{2} \times \frac{bc - ad}{ac - b^2}$

(iv) $(bc - ad)^2 = 4(ac - b^2)(bd - c^2)$.

8. Show that if α is a repeated root of $a_nx^n + \dots + a_1x + a_0 = 0$, then α is also a root of $na_nx^{n-1} + \dots + 2a_2 + a_1 = 0$. Hence, or otherwise, solve the equation $24x^4 - 20x^3 - 6x^2 + 9x - 2 = 0$, given that three of its four roots are identical.

9. Prove that if the polynomial equation $f(x) = 0$ has a double root then $f(r) = 0$ and $f'(r) = 0$. State and prove the converse proposition. Hence or otherwise,

(a) determine the constants A and B so that $Ax^{n+1} + Bx^n + 1$ is divisible by $(x - 1)^2$.

(b) prove that the polynomial equation $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} = 0$ cannot have a double root.

10. Find the multiple roots of the equation $x^4 + 12x^3 + 32x^2 - 24x + 4 = 0$.

11. Find the multiple roots of the equation $x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8 = 0$.

12. Show that the equation $x^n - a^n = 0 \quad (a \neq 0)$ cannot have repeated roots.

13. Show that the equation $x^n + nx^{n-1} + n(n-1)x^{n-2} + \dots + n! = 0$ cannot have equal roots.

14. Show that the equation $x^4 + px^2 + q = 0$ cannot have exactly three equal roots.

15. Find the value of b/a so that the equations $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ may have

(a) exactly one root in common;

(b) exactly two roots in common.

Find also the common root(s) in each of the above cases.

16. Given that $x^3 - x^2 + 6x + 24 = 0$, $x^2 - x + b = 0$ have a common root, find the value of b and the common root.

17. The equations $2x^3 + 5x^2 - 6x - 9 = 0$ and $3x^3 + 7x^2 - 11x - 15 = 0$ have two common roots. Find them.

18. The equations $x^3 - 2x^2 - 2x + 1 = 0$ and $x^4 - 7x^2 + 1 = 0$ have two common roots. Find them.

19. Determine the common roots of the equations $6x^3 + 7x^2 - x - 2 = 0$ and $6x^4 + 19x^3 + 17x^2 - 2x - 6 = 0$.

20. Solve the following equations each of which has repeated roots:

(a) $4x^3 - 12x^2 - 15x - 4 = 0$,

(b) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$,

(c) $16x^4 - 24x^2 + 16x - 3 = 0$.

21. **(a)** Prove the two equations : $x^2 + ax + b = 0$ and $x^2 + a'x + b' = 0$ has a common root iff $(b - b')^2 + (a - a')(ab' - a'b) = 0$

(b) Find the value of k so that the two equations:

$2x^2 - (k + 2)x + 12 = 0$ and $4x^2 - (3k - 2)x + 36 = 0$

has a common root. Find also the value of this common root.